Journal of Pure and Applied Algebra 213 (2009) 1777–1787

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

The Poincaré series of a simple complete ideal of a two-dimensional regular local ring

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ARTICLE INFO

Article history:
Received 2 December 2008
Received in revised form 28 December 2008
Available online 8 February 2009
Communicated by A.V. Geramita

MSC:
13H05
13B22

ABSTRACT

The aim of this work is to introduce both a classical and a motivic Poincaré series associated with a residually rational simple complete $\mathfrak{m}$-primary ideal $\wp$ of a two-dimensional regular local ring $(R, \mathfrak{m})$. We describe them in terms of the generators of the value semigroup of $\wp$, and compare them with the Poincaré series arising from a general element f for $\wp$.

1. Introduction

The purpose of this paper is to introduce both a classical and a motivic Poincaré series associated with a simple complete ideal $\wp$ of a regular local ring R of dimension 2. Galindo [1] treated the classical Poincaré series, i.e. the case where the residue field of R is algebraically closed. We prove—among other things—a corresponding result for the case of an infinite residue field—using other methods.

The paper is organized as follows. Section 2 contains a brief summary of the main concepts and results in the theory of simple complete ideals of two-dimensional regular local rings. Section 3 deals with the concept of the semigroup $\Gamma_\wp$ of $\wp$ and its Poincaré series $P_{\Gamma_\wp}(t)$. In Section 4 we introduce the notion of Poincaré series $P_\wp(t)$ of $\wp$, we show that it is a rational function, describe it in terms of a minimal system of generators of $\Gamma_\wp$ (cf. Theorem 4.9), and give its functional equation (cf. Proposition 4.10). Also, we express $P_\wp(t)$ as a rational function with a denominator which depends only on the multiplicity of $\wp$ (cf. Proposition 4.14), and prove in Corollary 4.15 that the following data are equivalent: the Poincaré series $P_\wp(t)$ and the multiplicity of $\wp$; the series $P_{\Gamma_\wp}(t)$; and the semigroup $\Gamma_\wp$. Section 5 establishes the relation between $P_\wp(t)$ and the Poincaré series of a general element f for $\wp$, say $\overline{P}_f(t)$. Finally, in Section 6 we introduce a motivic version of $P_\wp(t)$ and proceed with a similar study.

2. Generalities about simple ideals

For notations, proofs and further details used in this section, the reader should refer to [2], Chapter VII, or [3], Appendix 5.

2.1

Let R be a two-dimensional regular local ring with maximal ideal $\mathfrak{m}$, residue field k and quotient field K. We say—following Zariski—that an ideal of R is complete if it is an integrally closed ideal of R. A non-zero $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ of R is

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.01.001

called simple if it is not the product of two proper ideals of R. Zariski showed: every non-zero $\mathfrak{m}$-primary complete ideal of R can be written in a unique way as a product of simple complete $\mathfrak{m}$-primary ideals of R (cf. [3], App. 5, Theorem 3; also [2], Ch. VII, Theorem 4.17).

2.2

Let $\wp \neq \mathfrak{m}$ be a simple complete $\mathfrak{m}$-primary ideal of R (in the rest of this paper, such an ideal is, for the sake of simplicity, called a simple ideal). Such an ideal determines a sequence

$$R_0 := R \subsetneqq R_1 \subsetneqq R_2 \subsetneqq \cdots \subsetneqq R_h =: S \tag{1}$$

such that:

- $R_1, \ldots R_h$ are two-dimensional regular local rings with quotient field K,
- for each $j \in \{1, \ldots, h\}$, the ring R_j is a quadratic transform of R_{j-1},
- the residue field k_j of R_j is a finite extension of k whose degree will be denoted by $[R_j : R]$.

The sequence (1) is called the *quadratic sequence determined by* $\wp$. We say that $\wp$ is a simple ideal of rank h (cf. [2], Ch. VII, (5.1)).

We set $f_\wp := [R_h : R]$. If $f_\wp = 1$, then the ideal $\wp$ is said to be *residually rational*. We shall assume in the sequel that the ideal $\wp$ *is residually rational.*

2.3

The order function ord_S of S gives rise to a discrete valuation of K of rank 1 which will be denoted by $\nu_\wp = \nu$. We consider ν as a map $\nu: K \to \mathbb{Z} \cup \{\infty\}$. We have $\nu(x) = \infty$ only for $x = 0$. Let us denote by V the valuation ring of ν. Then V dominates R and, for $j \in \{1, \ldots, h\}$, V dominates R_j; the ring R_j is the only quadratic transform of R_{j-1} which is contained in V.

2.4

Let $\mathfrak{a} \neq (0)$ be an ideal of R. Now $\mathfrak{a}$ is finitely generated, and so $\nu(\mathfrak{a}) := \min\{\nu(z) \mid z \in \mathfrak{a} \setminus \{0\}\}$ is well-defined. Remember that a non-zero ideal $\mathfrak{a}$ of R is a ν-ideal if the following equivalent conditions are satisfied (see [2], B(6.12)):

(1) $\mathfrak{a}V \cap R = \mathfrak{a}$;
(2) $\mathfrak{a} = \{z \in R \mid \nu(z) \geq \nu(\mathfrak{a})\}$.

In particular, let $m \in \mathbb{N}_0$; then $\{z \in R \mid \nu(z) \geq m\}$ is a ν-ideal. Moreover, if $\mathfrak{a}$ is a ν-ideal, then the ν-ideal

$$\mathfrak{a}^+ := \{z \in R \mid \nu(z) \geq \nu(\mathfrak{a}) + 1\}$$

is called the *immediate ν-successor* of $\mathfrak{a}$.

Every ν-ideal of R is a complete $\mathfrak{m}$-primary ideal (cf. [3], Appendix 4, p. 353; also [2], B(6.16)(2)).

2.5

We consider the quadratic sequence (1) of 2.2 determined by $\wp$. For each $j \in \{0, \ldots, h\}$, there exists a unique simple ideal $\wp_j$ of R with $\wp_j^{R_j} = \mathfrak{m}(R_j)$, where $\wp_j^{R_j}$ is the transform of $\wp_j$ in R_j and $\mathfrak{m}(R_j)$ is the maximal ideal of R_j (cf. [2], Ch. VII, (5.3)). Note that $\wp_h = \wp$ and $\wp_0 = \mathfrak{m}$. We have a chain of simple ideals of R, namely

$$\mathfrak{m} = \wp_0 \supsetneqq \wp_1 \supsetneqq \cdots \supsetneqq \wp_h = \wp,$$

and the ideals $\wp_0, \ldots, \wp_{h-1}$ are ν-ideals of R (they are called the *predecessors* of $\wp$), hence complete; every power of $\wp$ is a ν-ideal and every $\mathfrak{m}$-primary ν-ideal of R is a product of these $h+1$ simple ν-ideals (cf. [2], Ch. VII, (5.3)).

From now on, we assume that *the residue field $k = R/\mathfrak{m}$ of R is infinite*. The results mentioned in Propositions 2.6 and 2.8 below are due to Noh (see Theorem 3.1 in [4]):

Proposition 2.6. *If $\mathfrak{a}$ is a ν-ideal, then, for every $n \in \mathbb{N}$, $\mathfrak{a}\wp^n$ is also a ν-ideal.*

Notation 2.7. For any complete $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R we denote by $\delta_\wp(\mathfrak{a})$ the non-negative integer such that $\wp^{\delta_\wp(\mathfrak{a})}$ divides $\mathfrak{a}$ but $\wp^{\delta_\wp(\mathfrak{a})+1}$ does not divide $\mathfrak{a}$.

Let us denote by $\lambda_R(\cdot)$ the length of an R-module. The number $\delta_\wp(\mathfrak{a})$ measures the following length:

Proposition 2.8. *Let $\mathfrak{a}$ be a non-zero proper ν-ideal of R. Let $\mathfrak{a}^+$ be its immediate ν-successor. Then one has*

$$\lambda_R\left(\mathfrak{a}/\mathfrak{a}^+\right) = \delta_\wp(\mathfrak{a}) + 1.$$

Note 2.9. Noh proved these results for k algebraically closed. The proof is based on the existence of a minimal reduction of $\wp$, which is still true if k is infinite (cf. [5]). It is not difficult to generalize Noh's proof using the fact that $\wp$ is residually rational (for a proof, cf. [6]).

3. The semigroup of a simple ideal

3.1

The discrete valuation ν of K arising from the order function of S (see 2.2) defines a semigroup

$$\Gamma_\wp := \{\nu(z) \mid z \in R \setminus \{0\}\} \subset \mathbb{N}_0,$$

which is called the *value semigroup* of $\wp$. Since K is the quotient field of V and R, there exist $a, b \in R\setminus\{0\}$ with $\nu(a)-\nu(b) = 1$, hence $\Gamma_\wp$ is a numerical semigroup, i.e., the greatest common divisor of its elements is 1; equivalently, it has a conductor $c_\wp$. Therefore $\Gamma_\wp$ is finitely generated. Notice that $c_\wp$ is the ν-value of the adjoint ideal of $\wp$ (cf. [7], Corollary (2.2.1)). This semigroup was studied by Noh [8] and Spivakovsky [9] for k an algebraically closed field, and by Lipman [7] and Greco and Kiyek [10,11] when $\wp$ is residually rational. The semigroup $\Gamma_\wp$ is symmetric (different proofs of this fact are given in [8,7] and [11]).

3.2

Furthermore, one can easily find a system of generators of $\Gamma_\wp$. We have $h = \operatorname{rank}(\wp)$, and we set $e := \nu(\wp)$. Let $\wp_0 = \mathfrak{m}, \wp_1, \dots, \wp_{h-1}$ be the predecessors of the ideal $\wp = \wp_h$. We set:

$$s_i := \nu(\wp_i) \quad \text{for } i \in \{0, \dots, h-1\},$$
$$s_h := e;$$

then $\Gamma_\wp$ is generated by $s_0, s_1, \dots, s_h$ (cf. 2.5). But this is not a minimal system of generators of $\Gamma_\wp$.

3.3

We use the results of [10]. From the Hamburger–Noether tableau of $\wp$, one can construct a system of generators $\{r_0, \dots, r_h\}$ of $\Gamma_\wp$, cf. loc. cit., (8.6), and from this the minimal system of generators $\{\rho_0, \dots, \rho_g\}$ of $\Gamma_\wp$, cf. loc. cit., (8.7). Setting

$$\theta_0 := \rho_0, \qquad \theta_{i+1} := \gcd(\rho_0, \dots, \rho_i) \quad \text{for } i \in \{1, \dots, g\},$$
$$n_i := \frac{\theta_i}{\theta_{i+1}} \quad \text{for } i \in \{1, \dots, g\},$$

we have

$$\theta_1 > \theta_2 > \cdots > \theta_{g+1} = 1, \qquad \rho_{i+1} > n_i\rho_i \quad \text{for } i \in \{1, \dots, g-1\}, \tag{$*$}$$

and every $\gamma \in \Gamma_\wp$ has a unique representation

$$\gamma = u_0\rho_0 + u_1\rho_1 + \cdots + u_g\rho_g$$

with non-negative integers $u_0, u_1, \dots, u_g$ and $u_i < n_i$ for $i \in \{1, \dots, g\}$, cf. loc. cit., (8.3). We say that $\Gamma_\wp$ is strictly generated by $\{\rho_0, \dots, \rho_g\}$.

Definition 3.4. Let Γ be a numerical semigroup, and let $\{\rho_0, \rho_1, \dots, \rho_g\}$ be the minimal system of generators of Γ. If $(*)$ in 3.3 is satisfied, then Γ is said to have *a strong growth*; such a semigroup is strictly generated by $\{\rho_0, \dots, \rho_g\}$ (in the sense of 3.3).

Proposition 3.5. (a) *We have*

1. *The semigroup of a simple ideal has a strong growth.*
2. *The semigroup of a plane irreducible algebroid curve over an algebraically closed field has a strong growth.*

(b) *A numerical semigroup with a strong growth can be realized as the value semigroup of a simple ideal, and also as the value semigroup of a plane irreducible algebroid curve over an algebraically closed field.*

Proof. The assertions regarding simple ideals follow from [10], (8.17) and (8.18); the first author of this paper takes the opportunity to correct an omission in that paper: We must also add the condition $\theta_1 > \cdots > \theta_{g+1} = 1$.

The assertions regarding plane irreducible algebroid curves follow from [12], Satz 2 and Satz 6. □

3.6

The conductor $c_\wp$ of $\Gamma_\wp$ can be written in terms of the minimal system of generators of $\Gamma_\wp$ as follows (cf. [11], Proposition 2.7):

$$c_\wp = -\rho_0 + 1 + \sum_{i=1}^{g} \rho_i(n_i - 1).$$

Definition 3.7. With notation as in Section 2, the ideal $\wp$ is said to be an s-ideal if S is proximate to R_i for some $i \in \{0, \ldots, h-2\}$, i.e., if S is satellite with respect to R.

Lemma 3.8. *The following properties hold:*

(i) $e \geq n_g \rho_g$*, and so* $\nu(\wp) \notin \{\rho_0, \ldots, \rho_g\}$ *(cf.* [10], (8.8)(2)*).*
(ii) $\wp$ *is an* s*-ideal if and only if* $e = n_g \rho_g$ *(cf.* [10], (9.6)*).*

Definition 3.9. Let Γ be a numerical semigroup. The Poincaré series associated with Γ is the formal power series

$$P_\Gamma(t) := \sum_{n \in \Gamma} t^n \in \mathbb{Z}[\![t]\!].$$

Proposition 3.10. *We assume that the semigroup* Γ *has a strong growth. Then, with notations as in Definition 3.4,* $P_\Gamma(t)$ *is a rational function; we have*

$$P_\Gamma(t) = \frac{1}{1 - t^{\rho_0}} \cdot \prod_{i=1}^{g} \frac{1 - t^{n_i \rho_i}}{1 - t^{\rho_i}}.$$

Proof. Since Γ has a strong growth, every $n \in \Gamma$ can be written uniquely as a linear combination $n = u_0\rho_0 + \cdots + u_g\rho_g$, where $u_0 \in \mathbb{N}_0$ and $0 \leq u_i < n_i$ for each $i \in \{1, \ldots, g\}$, and we get:

$$\begin{aligned}\sum_{n \in \Gamma_\wp} t^n &= \sum_{\substack{u_0 \geq 0 \\ 0 \leq u_i < n_i \text{ for } 1 \leq i \leq g}} t^{u_0\rho_0 + \cdots + u_g\rho_g} \\ &= \left(\sum_{u_0 \geq 0} t^{u_0\rho_0}\right) \cdot \left(\sum_{0 \leq u_1 < n_1} t^{u_1\rho_1}\right) \cdot \ldots \cdot \left(\sum_{0 \leq u_g < n_g} t^{u_g\rho_g}\right) \\ &= \frac{1}{1 - t^{\rho_0}} \cdot \frac{1 - t^{n_1\rho_1}}{1 - t^{\rho_1}} \cdot \ldots \cdot \frac{1 - t^{\rho_g \rho_l}}{1 - t^{\rho_g}}. \quad \square\end{aligned}$$

4. The Poincaré series of a simple ideal

4.1

For every $n \in \mathbb{N}_0$ we set

$$I_\wp(n) := \{z \in R \mid \nu(z) \geq n\};$$

the ideal $I_\wp(n)$ is a ν-ideal of R. Notice that if $n \in \Gamma_\wp$, then $\nu\left(I_\wp(n)\right) = n$. Since $\nu(\mathfrak{m}I_\wp(n)) \geq n + 1$, we have $\mathfrak{m}I_\wp(n) \subset I_\wp(n+1)$, hence the length

$$d_\wp(n) := \lambda_R\left(I_\wp(n)/I_\wp(n+1)\right) = \dim_k\left(I_\wp(n)/I_\wp(n+1)\right)$$

is finite. Furthermore, we set

$$\delta_\wp(n) := \delta_\wp(I_\wp(n)).$$

Lemma 4.2. *We have*

$$I_\wp(n+e) = \wp I_\wp(n), \tag{$*$}$$

$$\delta_\wp(n+e) = \delta_\wp(n) + 1. \tag{$**$}$$

Proof. $(*)$ follows from the fact that $\wp I_\wp(n)$ is a ν-ideal, by Proposition 2.6. $(**)$ is an immediate consequence of the definition of $\delta_\wp(n)$. $\square$

Definition 4.3. Let $\wp$ be a simple ideal of R. The formal power series

$$P_\wp(t) := \sum_{n\in\mathbb{N}_0} d_\wp(n)t^n \in \mathbb{Z}[\![t]\!]$$

is called the *Poincaré series* of $\wp$.

The coefficients of $P_\wp(t)$ can be described in terms of $\wp$.

Lemma 4.4. *For every $n \in \mathbb{N}_0$ we have*

$$d_\wp(n) = \begin{cases} 0 & \text{if } n \notin \Gamma_\wp, \\ \delta_\wp(n) + 1 & \text{if } n \in \Gamma_\wp. \end{cases}$$

Proof. If $n \notin \Gamma_\wp$, then for any $z \in I_\wp(n)$ one has that both $\nu(z) \geq n$ and $\nu(z) \geq n+1$, hence $I_\wp(n)/I_\wp(n+1) = 0$. If $n \in \Gamma_\wp$, then the ideal $I_\wp(n+1)$ is the immediate ν-successor of $I_\wp(n)$ and we apply Proposition 2.8 to obtain the result. □

4.5

Let $n \in \Gamma_\wp$. We can express n as a linear combination of the elements $s_0, \ldots, s_h$ with coefficients in $\mathbb{N}_0$. We choose a combination where the coefficient of s_h is maximal, i.e., we write

$$n = z_0(n)s_0 + \cdots + z_{h-1}(n)s_{h-1} + z_h(n)s_h$$

with $z_i(n) \in \mathbb{N}_0$ and $z_h(n)$ maximal.

Lemma 4.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$z_h(n) = \delta_\wp(n).$$

Proof. The element $n - z_h(n)e$ belongs to $\Gamma_\wp$. Then $I_\wp(n - z_h(n)e)$ is a ν-ideal with

$$\nu\left(I_\wp(n - z_h(n)e)\right) = n - z_h(n)e.$$

The ideal $\wp^{z_h(n)}I_\wp(n - z_h(n)e)$ is also a ν-ideal (by Proposition 2.6) with

$$\nu\left(\wp^{z_h(n)}I_\wp(n - z_h(n)e)\right) = n,$$

and so we have

$$\wp^{z_h(n)}I_\wp(n - z_h(n)e) = I_\wp(n). \tag{$*$}$$

On the other hand, we can write (cf. 2.5)

$$I_\wp(n - z_h(n)e) = \wp_0^{z'_0} \cdot \ldots \cdot \wp_h^{z'_h}$$

with $z'_0, \ldots, z'_h \in \mathbb{N}_0$. Then we get

$$\wp^{z_h(n)}I_\wp(n - z_h(n)e) = \wp_0^{z'_h} \cdot \ldots \cdot \wp_h^{z'_h + z_h(n)},$$

and so $n = z'_0s_0 + \cdots + (z'_h + z_h(n))s_h$. We obtain $z'_h = 0$ by our choice of $z_h(n)$. Therefore we get $z_h(n) = \delta_\wp(n)$. (To see this notice that $\delta_\wp(n) \geq z_h(n)$ from $(*)$, and writing $I_\wp(n) = \wp_0^{z''_0} \cdot \ldots \cdot \wp_{h-1}^{z''_{h-1}} \cdot \wp_h^{\delta_\wp(n)}$ then $n = z''_0s_0 + \cdots + z''_{h-1}s_{h-1} + \delta_\wp(n)e$, therefore $\delta_\wp(n) \leq z_h(n)$.) □

4.7

Let $\kappa \in \mathbb{N}_0$. We set

$$A_\kappa := \{\gamma \mid \gamma \in \Gamma_\wp, \gamma - \kappa e \in \Gamma_\wp\} \subseteq \mathbb{N}_0.$$

Note that $A_0 = \Gamma_\wp$. Let $\chi_{A_\kappa} : \mathbb{N}_0 \to \{0, 1\}$ be the characteristic function of A_κ, i.e., we have

$$\chi_{A_\kappa}(z) = \begin{cases} 1, & \text{if } z \in A_\kappa \\ 0, & \text{if } z \notin A_\kappa \end{cases} \quad \text{for every } z \in \mathbb{N}_0.$$

Proposition 4.8. *Let $n \in \Gamma_\wp$. Then we have*

$$\sum_{\kappa\in\mathbb{N}_0} \chi_{A_\kappa}(n) = \delta_\wp(n) + 1.$$

Proof. By Lemma 4.6, $\delta_\wp(n) = z_h(n)$, i.e., $n - ie \in \Gamma_\wp$ for $0 \le i \le \delta_\wp(n)$ and $n - (\delta_\wp(n)+1)e \notin \Gamma_\wp$, and the result follows. $\square$

We can describe the Poincaré series of $\wp$ in terms of the generators $\{\rho_0, \dots, \rho_g\}$ of the value semigroup of $\wp$:

Theorem 4.9. *Let $\wp$ be a simple ideal of R. Then:*
(1) *If $\wp$ is not an s-ideal, then*

$$P_\wp(t) = \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}} \cdot \frac{1}{1-t^e}.$$

(2) *If $\wp$ is an s-ideal, then*

$$P_\wp(t) = \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g-1} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}} \cdot \frac{1}{1-t^{\rho_g}}.$$

In particular, the Poincaré series $P_\wp(t)$ is a rational function.

Proof. By Lemma 4.4 and Proposition 4.8 we have:

$$\begin{aligned} P_\wp(t) &= \sum_{n\in\Gamma_\wp} (\delta_\wp(n)+1)t^n \\ &= \sum_{n\in\Gamma_\wp} \left(\sum_{\kappa=0}^{\infty} \chi_{A_\kappa}(n)\right) t^n. \end{aligned}$$

(a) For $\kappa \in \mathbb{N}_0$ we have

$$A_\kappa = \{n \mid n \in \Gamma_\wp, n - \kappa e \in \Gamma_\wp\} = \{n + \kappa e \mid n \in \Gamma_\wp\}.$$

(b) We have

$$\begin{aligned} P_\wp(t) &= \sum_{n\in\Gamma_\wp} \left(\sum_{\kappa=0}^{\infty} \chi_{A_\kappa}(n)\right) t^n \\ &= \sum_{n\in\Gamma_\wp} \left(\sum_{\substack{\kappa=0\\ n\ge\kappa e}}^{\infty} \chi_{A_\kappa}(n)\right) t^n \\ &= \sum_{\kappa=0}^{\infty} \left(\sum_{\substack{n\in\Gamma_\wp\\ n\ge\kappa e}} \chi_{A_\kappa}(n)t^n\right) \end{aligned}$$

(since for $\kappa \in \mathbb{N}_0$ the inner sum has order $\ge \kappa$). For $\kappa \in \mathbb{N}_0$ we have

$$\sum_{\substack{n\in\Gamma_\wp\\ n\ge\kappa e}} \chi_{A_\kappa}(n)t^n = \sum_{\substack{n\in\Gamma_\wp\\ n-\kappa e\in\Gamma_\wp}} t^n \stackrel{*}{=} \sum_{n\in\Gamma_\wp} t^{n+\kappa e}$$

(for the equality sign at $*$ use (a)). Therefore we obtain

$$\begin{aligned} P_\wp(t) &= \sum_{\kappa=0}^{\infty} \left(\sum_{n\in\Gamma_\wp} t^{n+\kappa e}\right) = \left(\sum_{\kappa=0}^{\infty} t^{\kappa e}\right) \cdot \left(\sum_{n\in\Gamma_\wp} t^n\right) \\ &= \frac{1}{1-t^e} \cdot \sum_{n\in\Gamma_\wp} t^n. \end{aligned}$$

Now we consider the second factor of this product. Following the reasonings in the proof of Proposition 3.10 we obtain

$$\begin{aligned} P_\wp(t) &= \frac{1}{1-t^e} \cdot \sum_{n\in\Gamma_\wp} t^n \\ &= \frac{1}{1-t^e} \cdot \frac{1}{1-t^{\rho_0}} \cdot \frac{1-t^{n_1\rho_1}}{1-t^{\rho_1}} \cdot \dots \cdot \frac{1-t^{n_g\rho_g}}{1-t^{\rho_g}}. \end{aligned} \tag{$\dagger$}$$

This is the assertion in (1). If $\wp$ is an s-ideal, then $e = n_g\rho_g$ and in ($\dagger$) the term $1 - t^{n_g\rho_g}$ cancels with $1 - t^e$, and we get the assertion in (2). $\square$

Notice that a geometric version (using divisorial valuations and algebraically closed residue fields) of this result was given in [1].

Proposition 4.10. *We have*

$$P_{\wp}(t^{-1}) = t^{e-c_{\wp}+1} \cdot P_{\wp}(t).$$

Proof. If $\wp$ is an s-ideal, then by Theorem 4.9(2) we obtain

$$\begin{aligned} P_{\wp}(t^{-1}) &= \frac{t^{\rho_0}}{t^{\rho_0}-1} \cdot \prod_{i=1}^{g-1} \frac{(t^{n_i\rho_i}-1)t^{\rho_i}}{(t^{\rho_i}-1)t^{n_i\rho_i}} \cdot \frac{t^{\rho_g}}{t^{\rho_g}-1} \\ &= t^{\rho_0+\sum_{i=1}^{g-1}\rho_i(1-n_i)+\rho_g} \cdot P_{\wp}(t). \end{aligned}$$

Since $e = n_g\rho_g$, we get

$$\begin{aligned} c_{\wp} &= -\rho_0 + 1 + \sum_{i=1}^{g} \rho_i(n_i-1) \\ &= -\rho_0 + 1 + \sum_{i=1}^{g-1} \rho_i(n_i-1) + e - \rho_g \end{aligned}$$

(the first equality is the formula of the conductor given in 3.6). Now

$$\rho_0 + \sum_{i=1}^{g-1} \rho_i(1-n_i) + \rho_g = e - c_{\wp} + 1;$$

this proves the assertion.

If $\wp$ is not an s-ideal, then by Theorem 4.9(1) we have

$$\begin{aligned} P_{\wp}(t^{-1}) &= \frac{t^{\rho_0}}{t^{\rho_0}-1} \cdot \prod_{i=1}^{g} \frac{(t^{n_i\rho_i}-1)t^{\rho_i}}{(t^{\rho_i}-1)t^{n_i\rho_i}} \cdot \frac{t^{e}}{t^{e}-1} \\ &= t^{\rho_0+\sum_{i=1}^{g}\rho_i(1-n_i)+e} \cdot P_{\wp}(t). \end{aligned}$$

Now

$$1 - c_{\wp} = \rho_0 + \sum_{i=1}^{g} \rho_i(1-n_i),$$

hence

$$\rho_0 + \sum_{i=1}^{g-1} \rho_i(1-n_i) + \rho_g = e - c_{\wp} + 1,$$

and the assertion follows. □

4.11

As the semigroup $\Gamma_{\wp}$ has a strong growth (cf. Proposition 3.5), we can calculate the Poincaré series $P_{\Gamma_{\wp}}(t)$ associated with $\Gamma_{\wp}$. In the two cases of the above Theorem 4.9 (namely, if $\wp$ is an s-ideal and $\wp$ is not an s-ideal) we obtain

$$P_{\wp}(t) = (1-t^{e})^{-1} \cdot P_{\Gamma_{\wp}}(t)$$

[remember that $e = n_g\rho_g$ if and only if $\wp$ is an s-ideal].

Corollary 4.12. *[Functional equation for $P_{\Gamma_{\wp}}(t)$]*

$$P_{\Gamma_{\wp}}(t^{-1}) = -t^{-c_{\wp}+1} \cdot P_{\Gamma_{\wp}}(t).$$

Proof. This is an easy consequence of 4.11 and Proposition 4.10. □

4.13

We set

$$A_\wp(t) := \sum_{\substack{n\in\Gamma_\wp \\ n<c_\wp}} t^n,$$

$$\Lambda_\wp(t) := t^{c_\wp} + (1-t)A_\wp(t);$$

then we have

$$P_{\Gamma_\wp}(t) = A_\wp(t) + \frac{t^{c_\wp}}{1-t}.$$

Proposition 4.14. *We can express the Poincaré series of* $\wp$ *as*

$$P_\wp(t) = \frac{\Lambda_\wp(t)}{(1-t)(1-t^e)}.$$

Proof. By 4.13 we get

$$P_{\Gamma_\wp}(t) = \frac{\Lambda_\wp(t)}{1-t};$$

applying now 4.11 we are done. □

Corollary 4.15. *The following data are equivalent:*

(1) *The Poincaré series* $P_\wp(t)$ *and the multiplicity* $e = \nu(\wp)$.
(2) *The Poincaré series* $P_{\Gamma_\wp}(t)$.
(3) *The value semigroup* $\Gamma_\wp$.

Proof. (1) ⇒ (2): We have (cf. 4.11)

$$P_{\Gamma_\wp}(t) = (1-t^e)P_\wp(t).$$

(2) ⇔ (3): This follows from the definition of $P_{\Gamma_\wp}(t)$ (cf. Definition 3.9).

(3) ⇒ (1): Given $\Gamma_\wp$, we construct a Hamburger–Noether Tableau as in [10], (8.19); then

$$e = q_1\theta_1 + \cdots + q_g\theta_g$$

by loc. cit., (7.12). □

5. General elements and Poincaré series

5.1

Let $f \in R$ be a general element for $\wp$ (for the definition of a *general element* we refer to [10], Definition (3.11)) and consider $\overline{R} := R/fR$. This ring is a one-dimensional local domain with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/fR$. As f is analytically irreducible (cf. [10], Corollary (4.10)), $\overline{R}$ is analytically irreducible, hence the integral closure $\overline{V}$ of $\overline{R}$ is a discrete valuation ring; let $\overline{\nu}$ be the discrete valuation of the quotient field of $\overline{R}$ determined by $\overline{V}$.

5.2

Let $\overline{\Gamma}_f$ be the semigroup $\overline{\nu}(\overline{R}\setminus\{0\})$. The semigroups $\Gamma_\wp$ and $\overline{\Gamma}_f$ coincide (cf. [10], Theorem 8.16).

5.3

Fix $n \in \mathbb{N}_0$, and define the ideal

$$\bar{J}(n) := \{z \in \overline{R} \mid \overline{\nu}(z) \geq n\}.$$

The length $\overline{d}(n) := \lambda_R\left(\bar{J}(n)/\bar{J}(n+1)\right)$ is finite (cf. [13], (3.1)); the formal power series

$$\overline{P}_f(t) := \sum_{n\in\mathbb{N}_0} \overline{d}(n)t^n \in \mathbb{Z}[\![t]\!]$$

is called the *Poincaré series associated with the general element* f for $\wp$.

We want to compare the Poincaré series $P_{\wp}(t)$ and $\overline{P}_f(t)$.

Notation 5.4. Let us consider the chain of quadratic transforms in 2.2 (1). Let g be a generator of the ideal $(fR)^S$ in S. Then $S/gS = \overline{V}$ (by [2], Ch. VIII, (8.5)). Both rings R and $\overline{R}$ have the same residue field k, and since $\wp$ is assumed to be residually rational, k is the residue field of $\overline{V}$. Hence we can apply [13], (3.2)(4) and (3.4), in order to show the following: We have

$$\overline{d}(n) = \begin{cases} 1 & \text{for every } n \in \overline{\Gamma}_f, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.5. *Let f be a general element for $\wp$. Then we have*

$$\overline{P}_f(t) = P_{\Gamma_{\wp}}(t).$$

In particular, the Poincaré series associated with the general element f depends only on the semigroup $\Gamma_{\wp}$.

Proof. This follows immediately from the last assertion in Notation 5.4, from 5.2 and from the definition of the power series $P_{\Gamma_{\wp}}(t)$ associated with $\Gamma_{\wp}$. □

6. A motivic version of the Poincaré series

6.1

Let k be a perfect field. Let $\mathcal{V}_k$ be the category of quasi-projective reduced schemes of finite type over k. The *Grothendieck ring* of $\mathcal{V}_k$, denoted by $K_0(\mathcal{V}_k)$, is defined to be the free Abelian group on isomorphism classes $[X]$ of schemes $X \in \mathcal{V}_k$, subject to the following relation (cf. [14,15]):

(1) $[X] = [X \setminus Z] + [Z]$ for any closed reduced subscheme Z of $X \in \mathcal{V}_k$,

and taking the fiber product as multiplication [1], i.e., one defines

(2) $[X_1] \cdot [X_2] := [X_1 \times_k X_2]$ for $X_1, X_2 \in \mathcal{V}_k$.

Remark 6.2. The neutral element of $K_0(\mathcal{V}_k)$ with respect to the addition will be denoted by 0; it is the class $[\emptyset]$ of the empty set.

The Grothendieck ring $K_0(\mathcal{V}_k)$ is a commutative ring with unit, the unit being the class of $\mathrm{Spec}(k)$.

6.3

Let $k[T]$ be the polynomial ring in one indeterminate T over the field k. The affine scheme $\mathrm{Spec}(k[T])$ over k is the affine line over k, which sometimes will be denoted by $\mathbb{A}^1_k$. The class of the affine line in $K_0(\mathcal{V}_k)$ is called the Lefschetz class of $K_0(\mathcal{V}_k)$, and is denoted by $\mathbb{L}$.

6.4

For every $n \in \mathbb{N}_0$ we define $\pi_n(\mathbb{L}) \in K_0(\mathcal{V}_k)$ by

$$\pi_n(\mathbb{L}) := \begin{cases} 0 & \text{if } n = 0, \\ 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^{n-1} & \text{otherwise.} \end{cases}$$

Remark 6.5. Notice that $\pi_n(\mathbb{L}) - \mathbb{L} \cdot \pi_{n-1}(\mathbb{L}) = 1$ for every $n \geq 1$.

6.6

Let V be a finite-dimensional k-vector space, and let $\mathbb{P}V$ be the projective space associated with V; then

$$[\mathbb{P}V] = \pi_{\dim_k(V)}(\mathbb{L}) \in K_0(\mathcal{V}_k).$$

Since the R-module $I_{\wp}(n)/I_{\wp}(n+1)$ is a finite-dimensional k-vector space, we can take its projectivization (under the action of $k \setminus \{0\}$), and also the class of this projective space in the Grothendieck ring. Inspired by the notion of generalized semigroup Poincaré series (see [16], Definition on pg. 201), we can define a Poincaré series of motivic nature in our context of simple ideals by setting

$$\widehat{P}_{\wp}(t) := \sum_{n \in \mathbb{N}_0} \left[\mathbb{P}\left(I_{\wp}(n)/I_{\wp}(n+1)\right)\right] t^n \in K_0(\mathcal{V}_k)[\![t]\!],$$

---

[1] Note that $X_1 \times_k X_2$ is reduced if X_1 and X_2 are reduced since k is a perfect field.

a formal power series with coefficients in the Grothendieck ring $K_0(\mathcal{V}_k)$. Since $\wp$ is residually rational, we have $d_\wp(n) = \dim_k \left(I_\wp(n)/I_\wp(n+1)\right)$ and therefore

$$\left[\mathbb{P}\left(I_\wp(n)/I_\wp(n+1)\right)\right] = \pi_{d_\wp(n)}(\mathbb{L}) \quad \text{for every } n \in \mathbb{N}_0.$$

This implies that the Poincaré series $\widehat{P}_\wp(t)$ can be expressed as:

$$\widehat{P}_\wp(t) = \sum_{n\in\Gamma_\wp} \pi_{d_\wp(n)}(\mathbb{L})t^n.$$

It can be also described by means of the generators $\{\rho_0, \ldots, \rho_g\}$ of the semigroup $\Gamma_\wp$, and therefore it can be related to the Poincaré series $P_{\Gamma_\wp}(t)$; this shall be the result of the next proposition.

Proposition 6.7. *We have*

$$\widehat{P}_\wp(t) = \frac{1}{1-\mathbb{L}t^e} \cdot \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}}.$$

Proof. By Lemma 4.2 and Proposition 2.8 we have:

$$\begin{aligned} \lambda_R\left(I_\wp(n)/I_\wp(n+1)\right) &= \delta_\wp(n)+1 \\ &= \delta_\wp(n+e) \\ &= \lambda_R\left(I_\wp(n+e)/I_\wp(n+e+1)\right)-1 \\ &= \lambda_R\left(\wp I_\wp(n)/\wp I_\wp(n+1)\right)-1. \end{aligned} \tag{$\dagger$}$$

Then we obtain:

$$\begin{aligned} (1-\mathbb{L}t^e)\widehat{P}_\wp(t) &= (1-\mathbb{L}t^e)\sum_{n\in\Gamma_\wp}\pi_{d_\wp(n)}(\mathbb{L})t^n \\ &= \sum_{n\in\Gamma_\wp}\pi_{d_\wp(n)}(\mathbb{L})t^n - \sum_{n\in\Gamma_\wp}\mathbb{L}\cdot\pi_{d_\wp(n)}(\mathbb{L})t^{n+e} \\ &= \sum_{n\in\Gamma_\wp}\left(\pi_{d_\wp(n)}(\mathbb{L}) - \mathbb{L}\cdot\pi_{d_\wp(n-e)}(\mathbb{L})\right)t^n \\ &\overset{*}{=} \sum_{n\in\Gamma_\wp}\left(\pi_{\delta_\wp(n)+1}(\mathbb{L}) - \mathbb{L}\cdot\pi_{\delta_\wp(n-e)+1}(\mathbb{L})\right)t^n \\ &= \sum_{n\in\Gamma_\wp}\left(\pi_{\delta_\wp(n)+1}(\mathbb{L}) - \mathbb{L}\cdot\pi_{\delta_\wp(n)}(\mathbb{L})\right)t^n \end{aligned}$$

(at $(*)$, we use Proposition 2.8 and $(\dagger)$).

By Remark 6.5, we have $\pi_{\delta_\wp(n)+1}(\mathbb{L}) - \mathbb{L}\cdot\pi_{\delta_\wp(n)}(\mathbb{L}) = 1$, and we get

$$(1-\mathbb{L}t^e)\widehat{P}_\wp(t) = \sum_{n\in\Gamma_\wp} t^n = P_{\Gamma_\wp}(t).$$

From Proposition 3.10 for $\Gamma = \Gamma_\wp$ we get the assertion. $\square$

As an immediate application of this result and Proposition 5.5 we obtain

Corollary 6.8.

$$\widehat{P}_\wp(t) = \frac{1}{1-\mathbb{L}t^e}\cdot P_{\Gamma_\wp}(t).$$

Both the relation between $P_\wp(t)$ and $\widehat{P}_\wp(t)$ and the functional equation of $\widehat{P}_\wp(t)$ is also clear:

Corollary 6.9. *We have*

$$(1-t^e)P_\wp(t) = (1-\mathbb{L}t^e)\widehat{P}_\wp(t),$$
$$\frac{1-\mathbb{L}t^e}{1-\mathbb{L}t^{-e}}\cdot\widehat{P}_\wp(t) = -t^{c_\wp-1}\cdot\widehat{P}_\wp(t^{-1}).$$

Acknowledgments

The second author was supported partially by the grant MEC MTM2007-64704, by Junta de CyL VA065A07, and by the grant DAAD-La Caixa. He is also thankful to the University of Paderborn for kind hospitality.

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